Electromagnetic solutions from vacuum fields

1975 J. Phys. A: Math. Gen. 8 1361

(http://iopscience.iop.org/0305-4470/8/8/020)

J. Phys. A: Math. Gen., Vol. 8, No. 8, 1975. Printed in Great Britain. © 1975

COMMENT

Electromagnetic solutions from vacuum fields

T Singh

Department of Mathematics, Gorakhpur University, Gorakhpur 273001, India

Received 10 October 1974, in final form 17 March 1975

Abstract. Applying Harrison's result to the empty space solutions of Taub and Das both due to the gravitational field of an infinite plane, we have obtained the corresponding solutions of the Einstein–Maxwell equations. These solutions represent fields due to infinite charged planes.

Harrison (1965) has proved an important result: 'Given any diagonalizable solution (or metric) of the vacuum field equations which is a function of no more than three variables, one can generate from this a solution of the coupled Einstein–Maxwell equations with nonzero electromagnetic field.' Mathematically we may rewrite this result in a very simple form. Suppose the metric

$$\mathrm{d}s^2 = -V^{-2}(\gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta)+V^2\,\mathrm{d}t^2 \tag{1}$$

with $\gamma_{\alpha\beta}$ and V as functions of x^1, x^2 and x^3, satisfies vacuum equations. Then the metric

$$\mathrm{d}s^2 = -[(V^2+1)^2/4\lambda^2V^2](\gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta)+[4\lambda^2V^2/(V^2+1)^2]\,\mathrm{d}t^2 \tag{2}$$

and the potential

$$C = \lambda(V^2-1)/(V^2+1) \tag{3}$$

satisfy the Einstein–Maxwell equations, λ being a constant. The potential C is connected to the magnetic potential A and the electric potential B through the relations

$$A = C\cos\alpha, \qquad B = C\sin\alpha, \tag{4}$$

α being a constant†. The components of the electromagnetic field tensor F_{ij} are given by

$$\begin{aligned} F^{\alpha\beta} &= (-g)^{-1/2}\epsilon^{\alpha\beta\gamma}A_{,\gamma} \\ F_{4\alpha} &= B_{,\alpha} \end{aligned} \tag{5}$$

where α, β, γ have values 1, 2, 3 and $\epsilon^{\alpha\beta\gamma}$ is the alternating three-index symbol.

In this note we have applied this result to the vacuum static plane symmetric solutions of Taub (1951) and the conformastat gravitational universe of Das (1971) to obtain the corresponding solutions of the Einstein–Maxwell equations. The possible physical interpretations of these solutions have also been given.

† This transformation may be looked upon as the duality rotation of Misner and Wheeler (1957). We may take the electromagnetic field to be purely electric or purely magnetic by choosing A or B to be zero respectively.

(i) *Static plane symmetric solution*

The static plane symmetric solution due to Taub (1951) is given by the metric

$$\begin{aligned} ds^2 &= -(k_1x+k_2)^{-1/2}(dx^2-dt^2)-(k_1x+k_2)(dy^2+dz^2) \\ &= -(k_1x+k_2)^{1/2}[(k_1x+k_2)^{-1}\,dx^2+(k_1x+k_2)^{1/2}(dy^2+dz^2)] \\ &\quad +(k_1x+k_2)^{-1/2}\,dt^2, \end{aligned} \tag{6}$$

where k_1, k_2 are constants. This solution has singularity at $x = -k_2/k_1$. Also this solution is due to the gravitational field of an infinite plane parallel to the (y, z) plane. By the use of Harrison's result the corresponding solution of the Einstein–Maxwell equations turns out to be

$$\begin{aligned} ds^2 = &-\{[1+(k_1x+k_2)^{1/2}]^2/4\lambda^2(k_1x+k_2)^{1/2}\}[(k_1x+k_2)^{-1}\,dx^2 \\ &+(k_1x+k_2)^{1/2}(dy^2+dz^2)]+\{4\lambda^2(k_1x+k_2)^{1/2}/[1+(k_1x+k_2)^{1/2}]^2\}\,dt^2 \end{aligned} \tag{7}$$

with the potential

$$C = \lambda[1-(k_1x+k_2)^{1/2}]/[1+(k_1x+k_2)^{1/2}]. \tag{8}$$

This solution is due to an infinite charged plane. Like metric (6) this solution also has singularity at $x = -k_2/k_1$. Thus we see that there is a one-to-one correspondence between the singularities in both cases.

With the help of (4), (5) and (8), on calculation, one can easily find that the purely electric and purely magnetic components of F_{ij} are respectively

$$F_{14} = \lambda k_1(k_1x+k_2)^{-1/2}[1+(k_1x+k_2)^{1/2}]^{-2}\sin\alpha \tag{9}$$

and

$$F_{23} = -\lambda k_1(k_1x+k_2)[1+(k_1x+k_2)^{1/2}]^{-2}\cos\alpha. \tag{10}$$

(ii) *Conformastat solution*

The empty space conformastat solution of Das (1971) is given by the metric

$$\begin{aligned} ds^2 &= -(1-mx)^4(dx^2+dy^2+dz^2)+(1-mx)^{-2}\,dt^2 \\ &= -(1-mx)^2[(1-mx)^2(dx^2+dy^2+dz^2)]+(1-mx)^{-2}\,dt^2, \end{aligned} \tag{11}$$

where m is a constant. This is due to the gravitational field of an infinite plane parallel to the (y, z) plane. The metric (11) has singularity at $x = 1/m$. The use of Harrison's result yields the solution of the Einstein–Maxwell equations representing an infinite charged plane and is given by

$$ds^2 = -\{[1+(1-mx)^2]^2/4\lambda^2\}(dx^2+dy^2+dz^2) + \{4\lambda^2(1-mx)^2/[1+(1-mx)^2]^2\}\,dt^2 \tag{12}$$

with the potential

$$C = \lambda[1-(1-mx)^2]/[1+(1-mx)^2]. \tag{13}$$

This solution also has a singularity at $x = 1/m$. Thus here again there is a one-to-one correspondence between the singularities of the metrics (11) and (12).

From (4), (5) and (13) the purely electric and purely magnetic components of F_{ij} are respectively

$$F_{14} = -4\lambda m(1-mx)[1+(1-mx)^2]^{-2} \sin\alpha \tag{14}$$

and

$$F_{23} = 4\lambda m(1-mx)^4[1+(1-mx)^2]^{-2} \cos\alpha. \tag{15}$$

Our conclusions about the singularities of the metrics (6), (7), (11) and (12) are consistent with Harrison's contention that there is a one-to-one correspondence between the singularities of the vacuum and electromagnetic solutions represented respectively by the metrics (1) and (2).

In conclusion, we at least hope that our investigation will lead to further understanding of the Einstein–Maxwell equations.

The author is thankful to Professor K B Lal for constant encouragement.

References

Das A 1971 *J. Math. Phys.* **12** 1136–42
Harrison B K 1965 *Phys. Rev.* **138** B488–94
Misner C W and Wheeler J A 1957 *Ann. Phys., NY* **2** 525
Taub A H 1951 *Ann. Math., NY* **53** 472–90